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A WEAK-TYPE INEQUALITY FOR DIFFERENTIALLY SUBORDINATE HARMONIC FUNCTIONS

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ABSTRACT. Assuming an extra condition, we decrease the constant in the sharp inequality of Burkholder $\mu(|v| \geq 1) \leq 2\|u\|_1$ for two harmonic functions u and v. That is, we prove the sharp weak-type inequality $\mu(|v| \geq 1) \leq K\|u\|_1$ under the assumptions that $|v(\xi)| \leq |u(\xi)|$, $|\nabla v| \leq |\nabla u|$ and the extra assumption that $\nabla u \cdot \nabla v = 0$. Here μ is the harmonic measure with respect to ξ and the constant K is the one found by Davis to be the best constant in Kolmogorov's weak-type inequality for conjugate functions.

Let D be a domain in $\mathbb{R}^n$ where n is a positive integer. Let D_0 be a bounded subdomain of D with $\partial D_0 \subseteq D$ and $\xi \in D_0$. Let μ be the harmonic measure on ∂D_0 with respect to ξ. Let K be the constant given by

$$K = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots}.$$

Theorem. *If u and v are harmonic functions on D such that*

$$\begin{array}{ll} \text{(i)} & |v(\xi)| \leq |u(\xi)|, \\ \text{(ii)} & |\nabla v| \leq |\nabla u| \quad \text{on } D, \\ \text{(iii)} & \nabla u \cdot \nabla v = 0 \quad \text{on } D, \end{array}$$

then

$$\mu(|v| \geq 1) \leq K \int_{\partial D_0} |u| \, d\mu.$$

Remarks. 1. The constant K was discovered by Davis [6]. He showed that K is the best constant in Kolmogorov's weak-type inequality for conjugate functions [9] or, equivalently, for the special case of the inequality above in which D is the open unit disk of $\mathbb{R}^2$, D_0 is an open disk with center 0 and radius $r < 1$, $\xi = 0$, $v(0) = 0$, and u and v are harmonic in D and satisfy the Cauchy-Riemann equations. Also, see Baernstein [2] for related sharp inequalities.

2. Dropping the classical conjugacy condition and working in $\mathbb{R}^n$, Burkholder [4] proved the sharp inequality

$$\mu(|u| + |v| \geq 1) \leq 2 \int_{\partial D_0} |u| \, d\mu$$

for harmonic functions u and v that satisfy the assumptions (i) and (ii) of the theorem. In fact, he proved his inequality for Hilbert-space valued u and v. In

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Example 13.1 of [5], he showed that 2 is the best constant even for the inequality $\mu(|v| \geq 1) \leq 2\|u\|_1$ where $\|u\|_p^p = \sup_{D_0} \int_{\partial D_0} |u|^p \, d\mu$.

3. Using (i) and (ii), Burkholder [4] also proved that $\|v\|_p \leq (p^* - 1)\|u\|_p$ for $1 < p < \infty$ where $p^* = \max\{p, p/(p-1)\}$. It is not yet known whether the constant $p^* - 1$ is best possible in this setting. However, using (i), (ii), and the extra assumption (iii), Bañuelos and Wang [3] proved the inequality $\|v\|_p \leq \cot(\pi/2p^*)\|u\|_p$ for $1 < p < \infty$. This is a sharp inequality since it is already sharp in the classical M. Riesz case [11] in which $v(0) = 0$ and v is the harmonic function conjugate to u on the open unit disk of the plane (see Pichorides [10] and the independent work of Brian Cole that is described in Gamelin [7]).

Outline of the proof of the theorem. Consider the function V on $\mathbb{R}^2$ given by

$$V(x,y) = \begin{cases} -K|x| & \text{if } |y| < 1, \\ 1 - K|x| & \text{if } |y| \geq 1. \end{cases}$$

We observe that

$$\mu(|v| \geq 1) - K \int_{\partial D_0} |u| \, d\mu = \int_{\partial D_0} V(u,v) \, d\mu.$$

The following lemma will be proved later:

Main Lemma. *There is a continuous function U on $\mathbb{R}^2$ such that*

(a) $V \leq U$ *on* $\mathbb{R}^2$,
(b) $U(u,v)$ *is superharmonic on* D,
(c) $U(x,y) \leq 0$ *if* $|x| \geq |y|$.

Then from (a) and (b) we get

$$\int_{\partial D_0} V(u,v) \, d\mu \leq \int_{\partial D_0} U(u,v) \, d\mu \leq U(u(\xi), v(\xi))$$

because μ is the harmonic measure on ∂D_0 with respect to ξ. Finally, (c) and the assumption (i) imply that $U(u(\xi), v(\xi)) \leq 0$, which proves the theorem.

Before we prove the lemma we define another function W on $\mathbb{R}^2$ and establish some properties of W. We will use basic facts about harmonic functions, which can be found in [1] and [8].

Put $H = \{(\alpha, \beta) : \beta > 0\}$, $S = \{(x,y) : |y| < 1\}$ and $S^+ = \{(x,y) \in S : x > 0\}$. Also, put $(x,y) = x + iy = z$, $\Im(x+iy) = y$, $(\alpha, \beta) = \alpha + i\beta = \zeta$, and define a function $\mathcal{W}$ on H by

$$\mathcal{W}(\alpha,\beta) = \mathcal{W}(\zeta) = \frac{2}{\pi^2} \int_{-\infty}^{\infty} \frac{\beta |\log|t||}{(\alpha - t)^2 + \beta^2} \, dt. \tag{1}$$

Observe that $\mathcal{W}$ is the harmonic function on H that vanishes as $\beta \to \infty$, and satisfies

$$\lim_{(\alpha,\beta) \to (t,0)} \mathcal{W}(\alpha,\beta) = \frac{2}{\pi} |\log|t|| \quad \text{if } t \neq 0.$$

Using $\pi^2/8 = \sum_{k=0}^{\infty}(2k+1)^{-2}$, we have that

$$\begin{aligned}\mathcal{W}(0,1) &= \frac{4}{\pi^2}\int_0^\infty \frac{|\log t|}{t^2+1}\,dt\\ &= \frac{4}{\pi^2}\int_{-\infty}^\infty \frac{|s|e^s}{e^{2s}+1}\,ds\\ &= \frac{8}{\pi^2}\int_0^\infty se^{-s}\sum_{k=0}^\infty(-e^{-2s})^k\,ds\\ &= \frac{8}{\pi^2}\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^2} = \frac{1}{K}.\end{aligned}$$

Consider the conformal map φ on S given by

$$\varphi(z) = ie^{\pi z/2} = \exp\frac{\pi}{2}(z+i).$$

Observe that $\varphi(i) = -1$, $\varphi(-i) = 1$, $\varphi(-\infty+iy) = 0$, $\varphi(\infty+iy) = \infty$ and $\varphi(0) = i$. Hence φ maps the strip S onto the upper half plane H. Define $W\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$W(x,y) = \begin{cases}|x| & \text{if } |y| \geq 1,\\ \mathcal{W}(\varphi(x,y)) & \text{if } |y| < 1,\end{cases}$$

and notice that the restriction of W to S is harmonic since this restriction is the real part of an analytic function. For $x_0 \in \mathbb{R}$ we have $\varphi(x_0, \pm 1) = \pm e^{\pi x_0/2} \neq 0$, thus

$$\lim_{\substack{(x,y)\to(x_0,\pm1)\\(x,y)\in S}} W(x,y) = \frac{2}{\pi}|\log|\varphi(x_0,\pm1)|\,| = |x_0| = W(x_0,\pm1).$$

Hence W is continuous on $\mathbb{R}^2$ as is the function U defined by

$$U(x,y) = 1 - KW(x,y) \quad \text{for } (x,y)\in\mathbb{R}^2.$$

Lemma 1. *If $(x,y)\in S$, then $W(x,y) = W(-x,y) = W(x,-y)$ and*

$$W_x(0,y) = W_y(x,0) = W_{xy}(x,0) = W_{xy}(0,y) = 0.$$

Proof. In (1) we use the change of variable $t = -s$ to get $\mathcal{W}(-\alpha,\beta) = \mathcal{W}(\alpha,\beta)$. Also, in the reformulation of $\mathcal{W}$

$$\mathcal{W}(\zeta) = \frac{4}{\pi^2}\Im\int_0^\infty \frac{\zeta|\log t|}{t^2-\zeta^2}\,dt$$

we use the change of variable $t = 1/s$ to get $\mathcal{W}(1/\bar\zeta) = \mathcal{W}(\zeta)$. With $\varphi(x,y) = \zeta = \alpha + i\beta$ we get $\varphi(-x,y) = 1/\bar\zeta$ and $\varphi(x,-y) = -\alpha + i\beta$. The symmetry of W and the rest of the lemma follow.

Lemma 2. $\lim_{\substack{x\to\infty\\(x,y)\in S}}[W(x,y) - x] = 0.$

Proof. $\varphi(x,y) = \zeta$ we have $x = \frac{2}{\pi}\log|\zeta|$, hence $x\to\infty$ if and only if $|\zeta|\to\infty$ and the lemma is equivalent to

$$\lim_{|\zeta|\to\infty}\left[\mathcal{W}(\zeta) - \frac{2}{\pi}\log|\zeta|\right] = 0. \tag{2}$$

On H, the harmonic function $\zeta \mapsto \frac{2}{\pi}\log|\zeta|$ can be represented by its Poisson integral. Therefore, by (1),

$$\begin{aligned}\mathcal{W}(\zeta) - \frac{2}{\pi}\log|\zeta| &= \frac{2}{\pi^2}\int_{-\infty}^{\infty}\left[\frac{\beta|\log|t||}{(\alpha-t)^2+\beta^2} - \frac{\beta\log|t|}{(\alpha-t)^2+\beta^2}\right]dt \\ &= \frac{4}{\pi^2}\int_{-1}^{1}\frac{\beta|\log|t||}{(\alpha-t)^2+\beta^2}\,dt \to 0 \quad \text{as } |\zeta|\to\infty\end{aligned}$$

which proves (2), hence the lemma.

Lemma 3. $\lim_{\substack{x\to\infty \\ (x,y)\in S}} W_{xx}(x,y) = \lim_{\substack{x\to\infty \\ (x,y)\in S}} W_{xy}(x,y) = 0.$

Proof. Consider the continuous function G on $\overline{S^+}$ given by $G(x,y) = W(x,y) - x$. Observe that G is harmonic on S^+ and $G(x,\pm 1) = 0$.

We consider a conformal map ψ given by $\psi(z)\sin(\frac{\pi}{2}iz) = -1$. One checks that $\psi(\frac{2}{\pi}\log(1+\sqrt{2})) = i$, $\psi(0) = \infty$, $\psi(-i) = -1$, $\psi(\infty) = 0$ and $\psi(i) = 1$. Thus S^+ is mapped onto H under ψ.

We define a harmonic function $F(\alpha,\beta)$ on H by $G = F\circ\psi$. For $|t| < 1$ we have

$$(3) \qquad \lim_{(\alpha,\beta)\to(t,0)} F(\alpha,\beta) = 0.$$

Indeed, we have from Lemma 2 that if $t = 0$, then

$$0 = \lim_{\substack{x\to\infty \\ (x,y)\in S}} G(x,y) = \lim_{(\alpha,\beta)\to(0,0)} F(x,y).$$

Also, for $t \neq 0$, since $\psi^{-1}(t,0) = (c,\pm 1)$ for some c and $G(c,\pm 1) = 0$, the limit (3) follows from the continuity of G.

Applying the Schwarz reflection principle, we see that the functions $F_\alpha(\alpha,\beta)$, $F_\beta(\alpha,\beta)$, $F_{\alpha\alpha}(\alpha,\beta)$, $F_{\alpha\beta}(\alpha,\beta)$ and $F_{\beta\beta}(\alpha,\beta)$ tend to certain limits as (α,β) tends to $(0,0)$.

Now from the basic identities

$$|\cos iz|^2 = \sinh^2 x + \cos^2 y \quad \text{and} \quad |\sin iz|^2 = \sinh^2 x + \sin^2 y$$

we observe that

$$\lim_{x\to\infty}\cos\frac{\pi}{2}iz = \lim_{x\to\infty}\sin\frac{\pi}{2}iz = \infty \quad \text{and} \quad \lim_{x\to\infty}\left|\tan\frac{\pi}{2}iz\right| = 1.$$

Differentiating $\psi(z)\sin(\frac{\pi}{2}iz) = -1$, we get

$$\begin{gathered}\psi'(z)\sin\frac{\pi}{2}iz + \frac{\pi}{2}i\psi(z)\cos\frac{\pi}{2}iz = 0, \\ \psi''(z)\sin\frac{\pi}{2}iz + \pi i\psi'(z)\cos\frac{\pi}{2}iz + \frac{\pi^2}{4}\psi(z)\sin\frac{\pi}{2}iz = 0.\end{gathered}$$

Hence, if $z = (x,y) \in S^+$ and $|z| \to \infty$, then $\lim\psi(z) = \lim\psi'(z) = \lim\psi''(z) = 0$.

Writing $\psi(x+iy) = \alpha(x,y) + i\beta(x,y)$, we see that as $(x,y)\in S^+$ and $x\to\infty$ all the functions α, β, α_x, β_x, α_{xx}, β_{xx} tend to 0 because $\psi' = \alpha_x + i\beta_x$ and $\psi'' = \alpha_{xx} + i\beta_{xx}$.

From the Cauchy-Riemann equations we have $\alpha_y = -\beta_x$ and $\beta_y = \alpha_x$, so $\alpha_{xy} = -\beta_{xx}$ and $\beta_{xy} = \alpha_{xx}$. Thus, using the chain rule and omitting the argument (x,y), we have

$$\begin{aligned} G_x &= \alpha_x F_\alpha(\alpha,\beta) + \beta_x F_\beta(\alpha,\beta),\\ G_{xx} &= (\alpha_x)^2 F_{\alpha\alpha}(\alpha,\beta) + (\beta_x)^2 F_{\beta\beta}(\alpha,\beta) + 2\alpha_x\beta_x F_{\alpha\beta}(\alpha,\beta)\\ &\quad + a_{xx}F_\alpha(\alpha,\beta) + \beta_{xx}F_\beta(\alpha,\beta),\\ G_{xy} &= -\alpha_x\beta_x F_{\alpha\alpha}(\alpha,\beta) + \alpha_x\beta_x F_{\beta\beta}(\alpha,\beta) + [(\alpha_x)^2 - (\beta_x)^2]F_{\alpha\beta}(\alpha,\beta)\\ &\quad - \beta_{xx}F_\alpha(\alpha,\beta) + \alpha_{xx}F_\beta(\alpha,\beta). \end{aligned}$$

It follows that

$$\lim_{\substack{x\to\infty\\(x,y)\in S^+}} G_{xx}(x,y) = \lim_{\substack{x\to\infty\\(x,y)\in S^+}} G_{xy}(x,y) = 0.$$

This proves the lemma because $G_{xx} = W_{xx}$ and $G_{xy} = W_{xy}$.

Lemma 4. *Consider the function A on H given by*

$$A(x,y) = \frac{1}{\pi}\int_{-1}^{1} \frac{y|t|}{(x-t)^2+y^2}\,dt.$$

Then we have

$$\liminf_{(x,y)\to(0,0)} A_{xx}(x,y) \geq 0 \quad \textit{and} \quad \limsup_{\substack{(x,y)\to(0,0)\\x>0}} A_{xy}(x,y) \leq 0.$$

Proof. Differentiating under the integral sign and then integrating by parts, we get

$$\begin{aligned} \pi A_x(x,y) &= \int_0^1 t\frac{\partial}{\partial t}\left[\frac{y}{(x+t)^2+y^2} - \frac{y}{(x-t)^2+y^2}\right]dt\\ &= \frac{y}{(x+1)^2+y^2} - \frac{y}{(x-1)^2+y^2}\\ &\quad - \int_0^1 \left[\frac{y}{(x+t)^2+y^2} - \frac{y}{(x-t)^2+y^2}\right]dt. \end{aligned}$$

Differentiating under the integral again, we get

$$\begin{aligned} \pi A_{xx}(x,y) &= -\frac{2(x+1)y}{[(x+1)^2+y^2]^2} + \frac{2(x-1)y}{[(x-1)^2+y^2]^2}\\ &\quad - \frac{y}{(x+1)^2+y^2} - \frac{y}{(x-1)^2+y^2} + \frac{2y}{x^2+y^2} \end{aligned}$$

and

$$\begin{aligned} \pi A_{xy}(x,y) &= \frac{(x+1)^2-y^2}{[(x+1)^2+y^2]^2} - \frac{(x-1)^2-y^2}{[(x-1)^2+y^2]^2}\\ &\quad + \frac{x-1}{(x-1)^2+y^2} + \frac{x+1}{(x+1)^2+y^2} - \frac{2x}{x^2+y^2}. \end{aligned}$$

Since $y > 0$ we have

$$\liminf_{(x,y)\to(0,0)} A_{xx}(x,y) = \frac{1}{\pi}\liminf_{(x,y)\to(0,0)} \frac{2y}{x^2+y^2} \geq 0.$$

Also,

$$\limsup_{\substack{(x,y)\to(0,0)\\x>0}} A_{xy}(x,y) = \frac{1}{\pi}\limsup_{\substack{(x,y)\to(0,0)\\x>0}} \left(-\frac{2x}{x^2+y^2}\right) \leq 0.$$

Lemma 5.

$$\liminf_{\substack{(x,y)\to(0,-1)\\(x,y)\in S}} W_{xx}(x,y) \geq 0 \quad \textit{and} \quad \limsup_{\substack{(x,y)\to(0,-1)\\(x,y)\in S\\x>0}} W_{xy}(x,y) \leq 0.$$

Proof. Let A be the function given in Lemma 4. Define $B(x,y)$ on S by

$$B(x,y) = W(x,y) - A(x,y+1).$$

Observe that B is harmonic on S and if $|x_0| < 1$, then

$$\lim_{(x,y)\to(x_0,-1)} B(x,y) = |x_0| - \lim_{(x,y)\to(x_0,0)} A(x,y) = 0.$$

Applying the Schwarz reflection principle we get a harmonic extension B^* of B over $S \cup \{(x,-1) : |x| < 1\} \cup \{(x,y) : x \in \mathbb{R}, -3 < y < -1\}$.

Note that $B^*(x,-1) = 0$ for $|x| < 1$. Thus $B^*_{xx}(0,-1) = 0$. Both W and A are symmetric with respect to y-axis, hence so is B^*. Thus $B^*_{xy}(0,-1) = 0$. From Lemma 4 and the limit

$$\lim_{(x,y)\to(0,-1)} B_{xx}(x,y) = B^*_{xx}(0,-1) = 0$$

we get

$$\liminf_{\substack{(x,y)\to(0,-1)\\(x,y)\in S}} W_{xx}(x,y) = \lim_{\substack{(x,y)\to(0,-1)\\(x,y)\in S}} B_{xx}(x,y) + \liminf_{\substack{(x,y)\to(0,0)\\(x,y)\in H}} A_{xx}(x,y) \geq 0.$$

The inequality about $\limsup W_{xy}$ is obtained similarly.

Lemma 6. *If* $x_0 > 0$*, then*

$$\lim_{\substack{(x,y)\to(x_0,-1)\\(x,y)\in S}} W_{xx}(x,y) = 0 \quad \textit{and} \quad \lim_{\substack{(x,y)\to(x_0,-1)\\(x,y)\in S}} W_{xy}(x,y) \leq 0.$$

Proof. Let $x_0 > 0$. Define a harmonic function C on S by $C(x,y) = W(x,y) - x$. Observe that for $x_0 \geq 0$ we have

$$\lim_{(x,y)\to(x_0,-1)} C(x,y) = 0.$$

We apply the Schwarz reflection principle to get a harmonic extension C^* of C over $S \cup \{(x,-1) : x > 0\} \cup \{(x,y) : x \in \mathbb{R}, -3 < y < -1\}$.

If $x > 0$, then $C^*(x,-1) = 0$, hence $C^*_{xx}(x,-1) = 0$ and

$$\lim_{\substack{(x,y)\to(x_0,-1)\\(x,y)\in S}} W_{xx}(x,y) = \lim_{(x,y)\to(x_0,-1)} C_{xx}(x,y) = C^*_{xx}(x_0,-1) = 0$$

which proves the first part of the lemma.

For the second part of the lemma we apply the Maximum Principle to C^*_{xy} over $\Omega = \{(x,y) : x > 0 \text{ and } -2 < y < 0\}$ to get $C^*_{xy}(x_0,-1) \leq 0$ which yields

$$\lim_{\substack{(x,y)\to(x_0,-1)\\(x,y)\in S}} W_{xy}(x,y) = \lim_{\substack{(x,y)\to(x_0,-1)\\(x,y)\in S}} C_{xy}(x,y) = C^*_{xy}(x_0,-1) \leq 0.$$

Now we will check the boundary conditions of the Maximum Principle. For $-3 < y < -1$ we have $C^*(x,y) = -C(x,-y-2)$, hence

$$C^*_{xy}(x,y) = C_{xy}(x,-y-2) = W_{xy}(x,-y-2).$$

From Lemma 1 we get $C^*_{xy}(0,y) = 0$ if $0 < |y+1| < 1$. Also, from Lemma 1 $W_{xy}(x,0) = 0$. Thus if $x_1 > 0$ and $|y_0+1| = 1$, then

$$\limsup_{\substack{(x,y)\to(x_1,y_0)\\(x,y)\in\Omega}} C^*_{xy}(x,y) = C^*_{xy}(x_1,y_0) = W_{xy}(x_1,0) = 0.$$

And using Lemma 3, we have

$$\limsup_{\substack{x\to\infty\\(x,y)\in\Omega}} C^*_{xy}(x,y) = \lim_{\substack{x\to\infty\\(x,y)\in S}} W_{xy}(x,y) = 0$$

because $C^*_{xy}(x,y) = W_{xy}(x,-y-2)$ for $-2 < y < -1$ and C^*_{xy} is continuous on Ω. Finally, the second part of Lemma 5 implies that

$$\limsup_{\substack{(x,y)\to(0,-1)\\(x,y)\in\Omega}} C^*_{xy}(x,y) = \limsup_{\substack{(x,y)\to(0,-1)\\(x,y)\in S\\x>0}} W_{xy}(x,y) \le 0.$$

This checks all the boundary conditions and finishes the proof of the lemma.

Lemma 7. *$W_{xx} \ge 0$ on S.*

Proof. We apply the Maximum Principle to the harmonic function $-W_{xx}$ over S. Observe, from Lemma 1, that $W_{xx}(-x,y) = W_{xx}(x,-y) = W_{xx}(x,y)$.

It remains to check the boundary conditions. The first part of Lemma 3 implies

$$\limsup_{\substack{|x|\to\infty\\(x,y)\in S}} [-W_{xx}(x,y)] = -\lim_{\substack{|x|\to\infty\\(x,y)\in S}} W_{xx}(x,y) = 0.$$

For $x_0 \ne 0$, the first part of Lemma 6 gives

$$\limsup_{\substack{(x,y)\to(x_0,\pm1)\\(x,y)\in S}} [-W_{xx}(x,y)] = -\lim_{\substack{(x,y)\to(x_0,-1)\\(x,y)\in S}} W_{xx}(x,y) = 0$$

Finally, the first part of Lemma 5 gives

$$\limsup_{\substack{(x,y)\to(0,\pm1)\\(x,y)\in S}} [-W_{xx}(x,y)] = -\liminf_{\substack{(x,y)\to(0,-1)\\(x,y)\in S}} W_{xx}(x,y) \le 0.$$

This proves the lemma.

Lemma 8. *$W_{xy} \le 0$ on $\Omega = \{(x,y) : x > 0$ and $-1 < y < 0\}$.*

Proof. Note that W_{xy} is harmonic on Ω and it is continuous on S. For $x_0 > 0$, Lemma 1 implies

$$\limsup_{\substack{(x,y)\to(x_0,0)\\(x,y)\in\Omega}} W_{xy}(x_0,0) = W_{xy}(x_0,0) = 0$$

and the second part of Lemma 6 implies

$$\limsup_{\substack{(x,y)\to(x_0,-1)\\(x,y)\in\Omega}} W_{xy}(x,y) = \lim_{\substack{(x,y)\to(x_0,-1)\\(x,y)\in S}} W_{xy}(x,y) \le 0.$$

Let $-1 < y_0 \le 0$. Lemma 1 gives

$$\limsup_{\substack{(x,y)\to(0,y_0)\\(x,y)\in\Omega}} W_{xy}(x,y) = W_{xy}(0,y_0) = 0.$$

Also, from the second part of Lemma 5,

$$\limsup_{\substack{(x,y)\to(0,-1)\\(x,y)\in\Omega}} W_{xy}(x,y) = \limsup_{\substack{(x,y)\to(0,-1)\\(x,y)\in S\\x>0}} W_{xy}(x,y) \le 0,$$

and from the second part of Lemma 3

$$\limsup_{\substack{x\to\infty\\(x,y)\in\Omega}} W_{xy}(x,y) = \lim_{\substack{x\to\infty\\(x,y)\in S}} W_{xy}(x,y) = 0.$$

Therefore we can apply the Maximum Principle and the lemma follows.

Proof of Main Lemma. We have defined the continuous function U on $\mathbb{R}^2$. It remains to show the properties (a), (b) and (c) of the function U.

Proof of (a). By the definitions we have $U(x,y) = V(x,y)$ if $|y| \ge 1$. Also, $W(0,0) = \mathcal{W}(\varphi(0,0)) = \mathcal{W}(0,1) = \frac{1}{K}$. Thus, if $|y| < 1$, then

$$U(x,y) = 1 - KW(x,y) = -K[W(x,y) - W(0,0)]. \tag{4}$$

Hence the property (a) follows if $-K|x| \le -K[W(x,y) - W(0,0)]$ on S. By the symmetry of W it suffices to show

$$E(x,y) \le 0 \quad \text{if} \quad (x,y) \in \overline{S^+} \tag{5}$$

where $E(x,y) = W(x,y) - W(0,0) - |x|$. Using Lemma 2 we have

$$\limsup_{\substack{x\to\infty\\(x,y)\in\overline{S^+}}} E(x,y) = -W(0,0) < 0.$$

Also $E(x,\pm1) = -W(0,0) < 0$ for $x \ge 0$. Since W is harmonic on S we have $W_{xx} + W_{yy} = 0$ thus $W_{yy} = -W_{xx} \le 0$ on S by Lemma 7. Hence, for $|y| < 1$, $E_{yy}(0,y) = W_{yy}(0,y) \le 0$ and $E(0,y)$ is a concave function on y. But $E_y(0,0) = W_y(0,0) = 0$ from Lemma 1. Thus $E(0,y) \le E(0,0) = 0$ for $|y| < 1$. Because E is continuous on $\overline{S^+}$ the Maximum Principle proves the inequality (5), hence the property (a).

Proof of (b). By (4) the property (b) becomes

$$W(u,v) \text{ is subharmonic on } D. \tag{6}$$

Arguing similarly as in the proof of (a) one gets

$$W(x,y) \ge |x| \quad \text{if} \quad |y| < 1. \tag{7}$$

Now we put $w = W(u,v)$ on D. When $|v| > 1$ clearly $w = |u|$ is subharmonic because u is harmonic. When $|v| < 1$, writing W_x for $W_x(u,v)$ etc., we have

$$\begin{aligned}\Delta w &= W_{xx}|\nabla u|^2 + W_{yy}|\nabla v|^2 + 2W_{xy}\nabla u\cdot\nabla v + W_x\Delta u + W_y\Delta v\\ &= W_{xx}(|\nabla u|^2 - |\nabla v|^2) \ge 0,\end{aligned}$$

hence w is subharmonic. In the above we used the assumptions (ii) and (iii), Lemma 7 and the harmonicity of u and v. When $|v| = 1$ at $\eta \in D$ we have, for all small $r > 0$, that

$$\operatorname{Avg}(w;\eta,r) \ge \operatorname{Avg}(|u|;\eta,r) \ge |u(\eta)| = w(\eta),$$

thus w is subharmonic at η. In the above we used the inequality (7). Also $\mathrm{Avg}(w;\eta,r)$ is the average of w over the ball $\{\lambda \in D : |\lambda - \eta| < r\}$ with respect to the Lebesgue measure in $\mathbb{R}^n$. This proves (6), hence (b).

Proof of (c). By (4) the property (c) of U follows from

$$W(x,y) \geq W(0,0) \quad \text{if} \quad |x| \geq |y|. \tag{8}$$

Let $I_0 = [0,\infty)$ and for $-1 \leq a < 0$, put $I_a = [0, -\frac{1}{a})$. Define Φ_a by $\Phi_a(t) = W(t,at)$ for $t \in I_a$. Then for t in the interior of I_a

$$\Phi_a'(t) = W_x(t,at) + aW_y(t,at)$$

and

$$\begin{aligned}\Phi_a''(t) &= W_{xx}(t,at) + a^2 W_{yy}(t,at) + 2aW_{xy}(t,at)\\ &= (1-a^2)W_{xx}(t,at) + 2aW_{xy}(t,at)\\ &\geq 0\end{aligned}$$

because W is harmonic, $W_{xx}(t,at) \geq 0$ by Lemma 7 and because $W_{xy}(t,at) \leq 0$ by Lemma 8. Observe that $\Phi_a'(0) = W_x(0,0) + aW_y(0,0) = 0$ by Lemma 1. Hence $\Phi_a(t) \geq \Phi_a(0)$ for $t \in I_a$. Thus $W(t,at) \geq W(0,0)$ if $-1 \leq a \leq 0$ and $t \in I_a$. But $\{(x,y) : x \geq -y \text{ and } -1 < y \leq 0\} = \{(t,at) : -1 \leq a \leq 0 \text{ and } t \in I_a\}$. Using the symmetry of W we have

$$W(x,y) \geq W(0,0) \quad \text{if} \quad |x| \geq |y| \quad \text{and} \quad |y| < 1.$$

Also, if $|x| \geq |y|$ and $|y| \geq 1$, then

$$W(x,y) = |x| \geq 1 > \frac{1}{K} = W(0,0).$$

This proves (8), hence (c).

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